Birational Zeta Functions

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§0: Motivation:

Def.: Let $f \in \mathbb{C}[x_1, \dots, x_n]$ be a non-constant polynomial. The motivic zeta function of f is:

$$Z_f^{mot}(T) := \sum_{m \geq 1} [\mathcal{X}_m(f)] \cdot \mathbb{L}^{-mn} T^m \in K(Var_{\mathbb{C}})[\mathbb{L}^{-1}][[T]]$$

$[\mathcal{X}_m(f)]$: space of m-jets along f

$\mathbb{L}$: $[\mathbb{A}^1_{\mathbb{C}}]$

Theorem (Denef-Loeser '98):

Let $\mu: Y \to \mathbb{A}^n_{\mathbb{C}}$ be a log resolution of $f^{-1}(0) = D$ and
$$\begin{cases} \mu^* D = \sum_{i \in S} N_i E_i \\ K_Y = \sum_{i \in S} (v_i - 1) E_i \end{cases}$$

Denote $E_I = \bigcap_{i \in I} E_i$. Then:

$$Z_f^{mot}(T) = \sum_{\emptyset \neq I \subseteq S} [E_I^{\circ}] \prod_{i \in I} \frac{\mathbb{L} - 1}{\mathbb{L}^{v_i} T^{-N_i} - 1}$$

Monodromy Conjecture (Igusa, ~70's)

$$\underbrace{\text{"pole of } Z_f^{mot}(T)\text{"}}_{\text{"mysterious cancellations"}} \rightsquigarrow \underbrace{\text{monodromy eigenvalue}}_{\sim \text{ well-understood (A'Campo)}}$$

In practice: smaller resolution

$\Downarrow$

less candidate poles

$\Downarrow$

less mysterious cancellations

e.g.: plane curves, hyperplane arrangements...

(Loeser '88), (Budur - Mustață - Teitler '11)

Idea: minimal ~~log~~ resolutions

Def.: A dlt modification is a proper birational morphism $\mu : (Y, \Delta) \to (X, D)$ with $\Delta = \mu_*^{-1} D + Ex(\mu)$, and (Y, Δ) minimal model of Y.

- **Xu ('16)**: "dlt motivic zeta function"

  But: heavily depends on the used dlt modification... (Nicaise-Potemans-Veys '24)

- **Our idea**: "Use a coarser coefficient ring."

§1: Birational zeta functions of dlt resolutions

<u>Def.</u>: • $\mathrm{Bir}^d_{\mathbb{C}} := \left\{ \begin{array}{l} \text{birational equivalence} \\ \text{classes of } d\text{-dim. alg. var.}/\mathbb{C} \end{array} \right\}$

• $\mathbb{Z}[\mathrm{Bir}^d_{\mathbb{C}}] :=$ free Abelian group generated by $\mathrm{Bir}^d_{\mathbb{C}}$

• $\mathbb{Z}[\mathrm{Bir}_{\mathbb{C}}] := \bigoplus_{d \in \mathbb{Z}_{\geq 0}} \mathbb{Z}[\mathrm{Bir}^d_{\mathbb{C}}]$

$\{Z_1\} \cdot \{Z_2\} = \{Z_1 \times_{\mathbb{C}} Z_2\}$

(Kontsevich-Tschinkel '19)
(Nicaise - Ottem '21)

<u>Notation</u>: • Complex variety Z of dimension d:

$\{Z\} \in \mathrm{Bir}^d_{\mathbb{C}}$

• $\mathbb{L} = \{\mathbb{A}^1_{\mathbb{C}}\} \in \mathrm{Bir}^1_{\mathbb{C}}$

• Throughout this talk:

X: smooth quasi-proj. alg. var. $/\mathbb{C}$

D: reduced non-zero divisor on X

<u>Def.</u>: Let $\mu : (Y, \Delta) \to (X, D)$ be a dlt resolution with $\mathrm{Supp}(\Delta) = \bigcup_{i \in S} E_i$ and $\begin{cases} \mu^* D = \sum_{i \in S} N_i E_i \\ K_{Y/X} = \sum_{i \in S} (\nu_i - 1) E_i \end{cases}$ $[[T]]$

The birational zeta function of μ is:

$$Z^{bir}_{X,D,\mu}(T) = \sum_{\emptyset \neq I \subset S} \{E_I\} \prod_{i \in I} \frac{\mathbb{L}}{\mathbb{L}^{\nu_i} T^{-N_i} - 1} \in \mathbb{Z}[\mathrm{Bir}_{\mathbb{C}}][\mathbb{L}^{-1}][[T]]$$

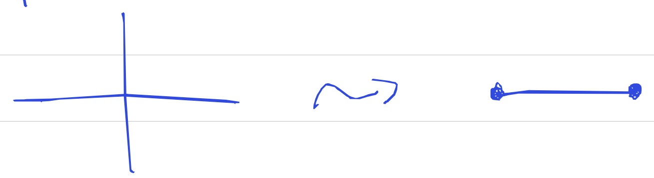
Proposition: Let μ be a dlt resolution of (X,D). Then $Z^{bir}_{X,D,\mu}(T)$ only depends on the crepant-birational equivalence class of μ.

Sketch\Idea of proof:

- Recall from (de Fernex-Kollár-Xu '17):

dlt pair $(Y,\Delta) \rightsquigarrow$ dual complex $\mathcal{D}(\Delta)$

e.g.:



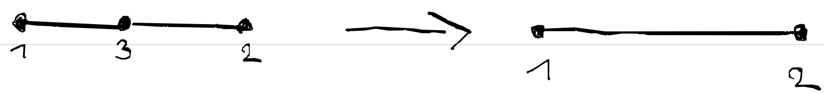
$\rightsquigarrow$ Follow proof of dFKX:

$$
\begin{tikzcd}
 & & (X,D) & & \\
 (Y_1,\Delta_1) \arrow[urr, "\mu_1"] & \rightsquigarrow \mathcal{D}(\Delta_1) & & (Y_2,\Delta_2) \arrow[ul, "\mu_2"'] & \\
 (Y_1',\Delta_1') \arrow[u, "e_1"] & \rightsquigarrow \mathcal{D}(\Delta_1'^{=1}) \arrow[u, equal] & & (Y_2',\Delta_2') \arrow[u, "e_2"'] & \\
 (Z_0,\Theta_0) \arrow[u, equal] & & & (Z_n,\Theta_n) \arrow[u, equal] & \\
 \cdots \arrow[u, "\pi_1"] & & & \cdots \arrow[u, "\pi_{n-1}"'] & \\
 & & (Z_m,\Theta_m) \arrow[ull, "\pi_m"] \arrow[ur, "\pi_{m+1}"'] & &
\end{tikzcd}
$$

Weak factorization theorem

$\rightsquigarrow$ After each π_i, $\mathcal{D}(\Theta_i^{=1})$ can only change by a stellar subdivision.

e.g.:



$$\cdot\ \{E_1\}\frac{\mathcal{L}}{\mathcal{L}^{v_1}T^{-N_1}-1}+\{E_2\}\frac{\mathcal{L}}{\mathcal{L}^{v_2}T^{-N_2}-1}+\frac{\mathcal{L}^2}{\mathcal{L}^{v_3}T^{-N_3}-1}$$

$$+\frac{\mathcal{L}^2}{(\mathcal{L}^{v_1}T^{-N_1}-1)(\mathcal{L}^{v_3}T^{-N_3}-1)}+\frac{\mathcal{L}^2}{(\mathcal{L}^{v_2}T^{-N_2}-1)(\mathcal{L}^{v_3}T^{-N_3}-1)}$$

$$=\frac{\mathcal{L}^2}{\mathcal{L}^{v_3}T^{-N_3}-1}\left(1+\frac{1}{\mathcal{L}^{v_1}T^{-N_1}-1}+\frac{1}{\mathcal{L}^{v_2}T^{-N_2}-1}\right)$$

$$=\frac{\mathcal{L}^2}{\mathcal{L}^{v_3}T^{-N_3}-1}\cdot\frac{\mathcal{L}^{v_1+v_2}T^{-N_1-N_2}-1}{(\mathcal{L}^{v_1}T^{-N_1}-1)(\mathcal{L}^{v_2}T^{-N_2}-1)}$$

$$=\frac{\mathcal{L}^2}{(\mathcal{L}^{v_1}T^{-N_1}-1)(\mathcal{L}^{v_2}T^{-N_2}-1)}$$

§3: Contact loci and dlt modifications

Def. Let $m \geq 1$.

- $\mathcal{L}_m(X) :=$ "space of m-jets on X"
  $= \left\{ \begin{array}{l} \text{morphisms of } \mathbb{C}\text{-schemes} \\ \operatorname{Spec}\left(\frac{\mathbb{C}[t]}{(t^{m+1})}\right) \to X \end{array} \right\}$

- $\mathcal{X}_m = \mathcal{X}_m(X, D)$
  $=$ "m-contact locus of (X,D)"
  $= \{ \gamma \in \mathcal{L}_m(X) \mid \operatorname{ord}_\gamma(D) = m \}$

- m-valuation of (X,D)
  $=$ divisorial valuation on $\mathbb{C}(X)$ induced by a prime divisor E on birational modification $\mu : Y \to X$ s.t. $\mu(E) \subseteq D$ and $\operatorname{ord}_E(D) \mid m$

$\cup|$

- Essential m-valuation of (X,D)
  $=$ m-valuation with center a prime divisor on an m-separating log resolution of (X,D)

  $E_i \cap E_j \neq \emptyset \Rightarrow N_i + N_j > m$

$\cup|$

- Dlt m-valuation of (X,D)
  $=$ m-valuation with center a prime divisor on a proj. dlt modification of (X,D)

(Budur, de la Bodega, de Lorenzo-Poza, Fernández de Bobadilla, Pełka '24)

- Essential m-valuations of (X, D)

$\updownarrow$

Irreducible components of $\mathcal{X}_m$

$\rightsquigarrow \mathcal{X}_m^{dlt} :=$ union of irred. comp. of $\mathcal{X}_m$ induced by dlt m-valuations

Theorem: Let $\mu : (Y, \Delta) \to (X, D)$ be a dlt modification. Then

$$Z^{bir}_{X,D,\mu}(T) = \sum_{l \geq 1} \{\mathcal{X}_l^{dlt}\} \cdot \mathbb{L}^{-ln} T^l$$

Sketch \ Idea of proof:

Strategy: Show for every $m \geq 1$ that

$$Z^{bir}_{X,D,\mu}(T) \equiv \sum_{l \geq 1} \{\mathcal{X}_l^{dlt}\} \mathbb{L}^{-ln} T^l \mod T^{m+1}$$

- LHS: does not depend on the chosen dlt modification (see earlier)

- Fix $m \geq 1$ and let $\mu'_m : (Y'_m, \Delta'_m) \to (X, D)$ be an m-separating log resolution

$\to$ Run the MMP over (X, D)

$$\begin{tikzcd} (Y'_m, \Delta'_m) \arrow[rr, dashed] \arrow[dr, "\mu'_m"'] & & (Y_m, \Delta_m) \arrow[dl, "\mu_m"] \\ & (X, D) & \end{tikzcd}$$

$\to$ Take $\mu = \mu_m$

- Irred. comp. $\mathcal{X}_{l,i} \subseteq \mathcal{X}_l^{dlt}$

$\iff$ prime divisor E_i on Y with $N_i | l$

and $\{\mathcal{X}_{l,i}\} = \{E_i\} \mathbb{L}^{l(n - \nu_i/N_i) + 1}$

- $\begin{cases} \cdot\ m\text{-separating log resolution} \\ \cdot\ \text{collapse: } \mathcal{D}(\Delta'_m) \to \mathcal{D}(\Delta_m) \end{cases}$

$\rightsquigarrow$ Only strata of codim. 1 contribute to $Z^{bir}_{X,D,\mu_m}(T) \mod T^{m+1}$

<u>Proposition</u>: $Z^{bir}_{X,D}(T) = \sum_{E \text{ div. val.}} \frac{[E]\,\mathbb{L}}{\mathbb{L}^{\nu_E} T^{-N_E} - 1}$

<u>Remark</u>: Berkovich geometric interpretation of $Z^{bir}_{X,D}$:

- Dual complex embeds into Berkovich generic fibre of formal t-adic completion of X along D.

$\rightsquigarrow$ "Dual complex lives in a $\mathbb{C}((t))$-analytic space."

- Birational types: $x \in \mathcal{D}(\Delta) \rightsquigarrow \mathcal{H}(x) \rightsquigarrow \widehat{\mathcal{H}(x)}$
- N_E: ramification degree of $\mathcal{H}(x)/\mathbb{C}((t))$
- ν_{E}/N_E: weight function

§ 4: Poles:

Question: (Birational Monodromy Conjecture):

"pole of $Z^{bir}_{X,D}(T)$" $\overset{?}{\Longrightarrow}$ monodromy eigenvalue

Question: "pole of $Z^{bir}_{f}(T)$" $\Longrightarrow$ "pole of $Z^{mot}_{f}(T)$"

Hold for:
- cones over smooth proj. hypersurfaces
- hyperplane arrangements
- plane curves

Theorem: For plane curves:

$\mathbb{L}^{-s_0}$ pole of $Z^{bir}_{f,a}$ $\Longrightarrow$ s_0 pole of $Z^{top}_{f,a}$

$\mathbb{L}^{-s_0}$ " " " $\Longleftarrow$ " " " "

<u>Remark</u>: $\mathbb{Z}[\mathrm{Bir}_{\mathbb{C}}]$ has zero divisors

$\rightsquigarrow$ Definition of poles is subtle/tricky

$\rightsquigarrow$ Define "rational zeta function by $\{Z\} \mapsto \mathbb{L}^{\dim Z}$

<u>e.g.</u>: Analog of Veys's conjecture for topological zeta function:

$T = \mathbb{L}^{\mathrm{lct}_x(X,D)}$ is always a pole of $Z^{\mathrm{rat}}_{f,x,\mu}(T)$ and the only possible one of order $\dim X$.